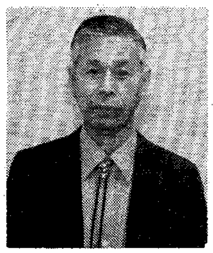

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Short Papers

An Accurate Approximation of the Impedance of a Circular Cylinder Concentric with an External Square Tube

H. J. RIBLET, FELLOW, IEEE

Abstract —**The problem of determining the characteristic impedance of a concentric coaxial transmission line having a circular inner conductor and a square outer conductor is reexamined. The Green's function for a rectangle is used to determine the geometrical capacitance of a series of structures ranging from 1–46 Ω with an error less than 10^{-5}. The method of analysis is illustrated in detail for the 1-Ω case. The results are presented in terms of the "outer shield factor" R_{eff}, which is defined as the ratio of the diameter of an outer circle, having the same capacitance as the outer square, to the side of the outer square. Values of this ratio are tabulated for impedances ranging from 1–46 Ω. These values are also plotted on a curve which can be read with an error of the order of 0.02 Ω for impedances greater than 3 Ω.**

I. Introduction

The determination of the characteristic impedance of the concentric coaxial line in which the outer conductor is a square and the inner conductor is a circle has been the subject of numerous treatments [1]–[16] appearing during the past forty years. In his discussion of this problem, Cohn [10] suggested that additional data between 30 and 2 Ω would be useful. This short paper provides this information.

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The treatment of this problem by Frankel [1], Oberhettinger and Magnus [2, pp. 75–78], and later by Laura and Luisoni [13], [14], is one in which the potential problem is solved exactly in a doubly connected region in which the outer conductor is a square, while the inner conductor is a four-lobed curve which approaches a circle ever more closely as its size decreases. Each circle internal to and concentric with the square has the same potential at eight equi-spaced points on its circumference. This potential function, except for an additive constant, is the Green's function [2, p. 36] for the square which has a logarithmic singularity at its center.

In this paper, a potential function is constructed, which is nearly constant on the outer circumference of the inner conductor, by suitably combining a number of Green's functions for the outer square whose logarithmic singularities are all inside of this circle.

II. The Green's Function

Fig. 1 shows an infinite lattice of positive and negative line charges whose logarithmic potential is zero along the boundary of a rectangle of width $2a$ and height $2b$ centered at the origin. This follows from the fact that, for every negative charge on one side of a boundary, there is an equal positive charge mirrored in it on the opposite side.

Consider for a moment the point Z' which is inside the rectangle in question. It determines a doubly infinite lattice of line charges which differ from it in location by integral multiples of $4a$ in the horizontal direction and by integral multiples of $4b$ in the vertical direction. Similar remarks can be made about the other three line charges shown in the upper right-hand quadrant

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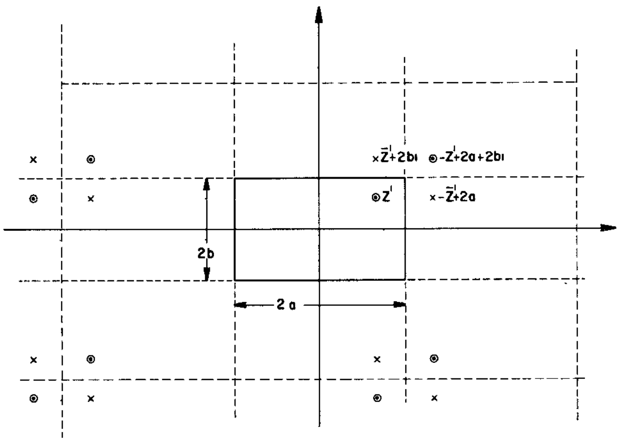

Fig. 1. Lattice of line changes.

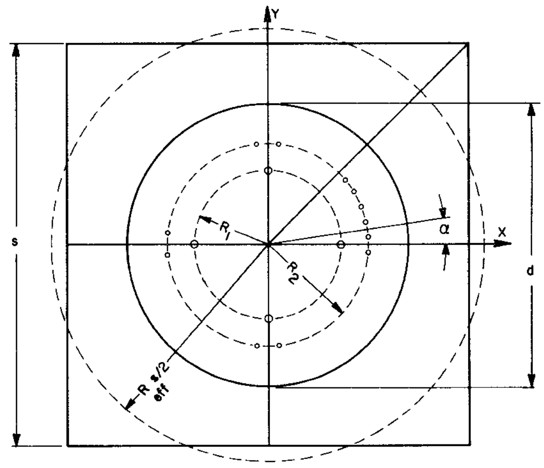

Fig. 2 Circle concentric with an external.

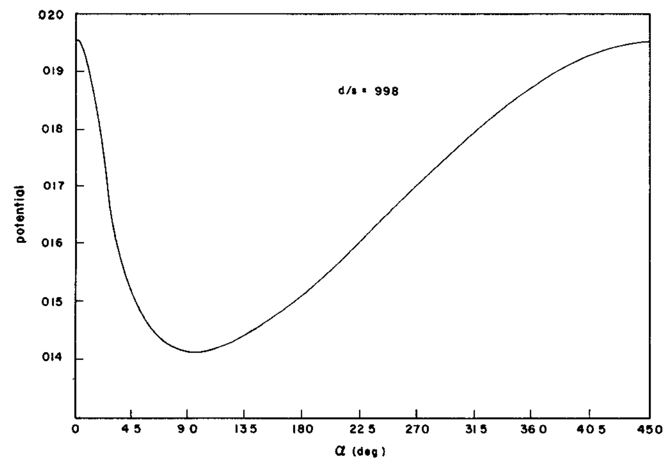

Fig 3. First approximation of potential.

of Fig. 1. The logarithmic potential due to the infinite lattice determined by Z' is well known and is given by Oberhettinger and Magnus [2, pp. 66–68]

$$U=-AR_e\ln\left(\theta_1\left(\frac{Z-Z'}{4a}\right)\right). \tag{1}$$

Here A is the coefficient of the logarithmic singularity at Z', and $\theta_1(x)$ is the Jacobi theta function defined in [17, p. 231] where $q=\exp(-\pi a/b)$. The logarithmic potential due to the four sets of infinite lattices determined by $Z', \bar{Z}'+2bi, -\bar{Z}'+2a$, and $-Z'+2a+2bi$ is given by

$$U=-AR_e\left\{\ln\left(\theta_1\left(\frac{Z-Z'}{4a}\right)\right)-\ln\left(\theta_1\left(\frac{Z-\bar{Z}'-2bi}{4a}\right)\right)-\ln\left(\theta_1\left(\frac{Z+\bar{Z}'-2a}{4a}\right)\right)+\ln\left(\theta_1\left(\frac{Z+Z'-2a-2bi}{4a}\right)\right)\right\}. \tag{2}$$

This function is the Green's function of the rectangle for the source point Z' since it can be shown that it has a value of zero on the boundary of the rectangle and satisfies Laplace's equation in the interior of the rectangle except at the point Z', where it has a logarithmic singularity.

For numerical computations, it is possible to expand $\theta_1(z)$ in a rapidly converging series in terms of q. If

$$\theta_1'(z)=\sin(\pi z)-q^2\sin(3\pi z)+q^6\sin(5\pi z)+\cdots \tag{3}$$

then $\theta_1(z)=2q^{\frac{1}{4}}\theta_1'(z)$; and because of its special form, the $\theta_1(z)$ of (2) can be replaced by $\theta_1'(z)$. Equation (3) converges with great rapidity since the rectangle can always be oriented so that $a/b \leqslant 1$. Then $q<0.044$. In evaluating the Green's function of (2), the arguments of the theta functions are complex, in general; but, as seen from (3), these complex numbers appear only in the arguments of the sine functions. This means that the real and imaginary parts of the theta function depend only on the addition formula for the sine function. Thus the real and imaginary parts of $\theta_1'(x+jy)$ in (3) are each rapidly converging power series in q whose coefficients are well-known elementary functions of x and y. As a consequence, the programming of (2) on a digital computer is a simple matter.

III. The First Approximation

Consider Fig. 2 where the problem is the determination of the total capacitance of the coaxial structure with a square outer conductor of side s, while the inner conductor is a circle having a diameter d. If four line charges are placed on the coordinate axes, as shown by the four smaller circles, each a suitable distance R_1 from the center, it will be found that the potential on the circle is the same at 45° as it is at 0°. In fact, Fig. 3 is a plot of the potential obtained by summing the Green's functions associated with the four line charges in the case when $d/s=0.998$. Here $R_1=0.948486$, while the value of the charge at the four points was normalized so that the characteristic impedance Z_0 of the coaxial line is given by the expression

$$Z_0=\frac{\eta}{2\pi}U. \tag{4}$$

Here η is the characteristic impedance of space and U is the potential.

If a conducting cylinder is placed to coincide with the circle on which the potential of Fig. 3 was found, its potential may be assumed to fall somewhere between the maximum on the curve, 0.01951, and the minimum, 0.01415. Then the characteristic impedance of a coaxial line, for which $d/s=0.998$, is given by (0.01683 ± 0.00268) 60 Ω = 1.10Ω ± 0.16 Ω if $\eta/2\pi$ is replaced by its approximate value, 60 Ω.

The radius of the external circle, as shown in Fig. 2, by which the outer conducting square may be replaced to give the same characteristic impedance, is readily found to exceed the side of the square by a factor of 1.0149, if the average value of the potential given above is assumed to be the exact value. This factor will be referred to as the "shield factor" of the square, following Wheeler [3, p. 1401]. It is appropriate to observe here that the limiting value of this quantity as the diameter of the inner conductor approaches zero was given by Frankel [1, (2)] approximately as 1.079 and later exactly by Oberhettinger and Magnus [2, p. 76] as $2/(K(k)(1+k))$ with $k=\tan^2(\pi/8)$.[1]

IV. The Improved Approximation

The deviation of the potential, on the circle of interest, from a constant value is relatively small even for an extreme case, as shown in Fig. 3, so it seems reasonable to fill in the low points in the potential by means of additional line changes uniformly spaced on a slightly smaller circle. How this was done for the $d/s = 0.7$ case is shown in Fig. 2. A total of 40 additional line changes, indicated by the small circles, were placed on a circle of radius, $R_2 = 0.5$. They were equally spaced and oriented so that they straddled the symmetry lines of the square. They were grouped into five sets of line charges. Each set contained eight members with the same charge positioned symmetrically about the symmetry lines of the square. One of these sets, the one containing the line charges nearest the x-axis, is shown in detail in Fig. 2. Because of the assumed symmetry, even if the sets have different charges, the potential distribution on the segment of the conducting circle of diameter d, lying between $0°$ and $45°$, will be typical of the potential distribution on the other seven similar segments. Now the potential at any point in the square is the sum of the potentials of the five independent sets of line charges plus the potential due the original set of four line charges. By selecting the relative magnitude of the charges of these six sets, it is possible to obtain a potential which has the same value at six equi-spaced points on this segment of the conducting circle at the angular positions, $0, \pi/20, \cdots \pi/4$. This potential, which is the same at forty equi-spaced points on a circle, has the value, 0.43114, to within one digit in the last place everywhere on the circle.

As the inner conductor approaches contact with the square, additional line charges on the circle of radius R_2 are required to achieve the same accuracy. In the extreme case considered here, when $d/s = 0.998$, twenty line charges were placed in each $45°$ sector. Then, with $R_2 = 0.9428$, a potential constant to within 6×10^{-7} was obtained. A plot of this potential, in one sector, is given in Fig. 4. The rapid oscillations occur within $18°$ of the point of contact. A value of 0.016600 is certainly correct to within one digit in the last place.

Calculations of this kind were made for values of the ratio of the diameter d of the internal circle to the side s of the external square ranging from 0.5 to 0.998. These results are summarized, most usefully perhaps, in Fig. 5 which plots the shield factor of the square R_{eff} versus d/s. It is felt, on the basis of trial, that this curve can be read to an accuracy of ± 0.0004 for values of d/s as large as 0.98. This corresponds to an error of about 0.02 Ω for all smaller values of d/s. Table I gives values of R_{eff} which are believed to be accurate to one digit in the last place, at least.

V. Comments

Of course, given d/s, the characteristics impedance of the structure is given by the formula

$$Z_0 = \frac{\eta}{2\pi}\ln\left(\frac{R_{\text{eff}}\cdot s}{d}\right) \tag{5}$$

[1] This equals 1.07871 to six places.

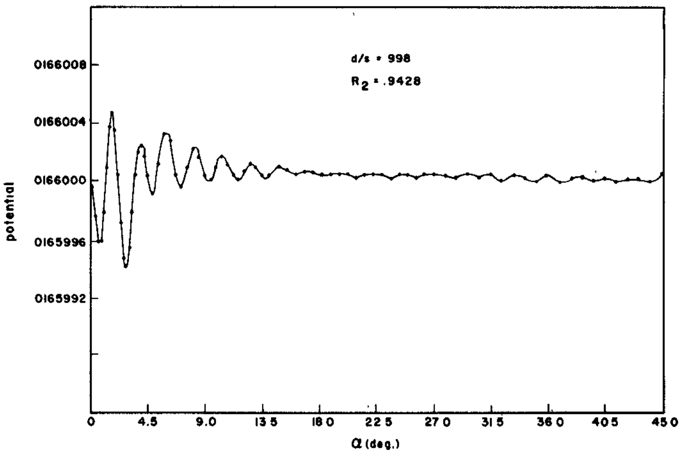

Fig. 4. Improved approximation of potential.

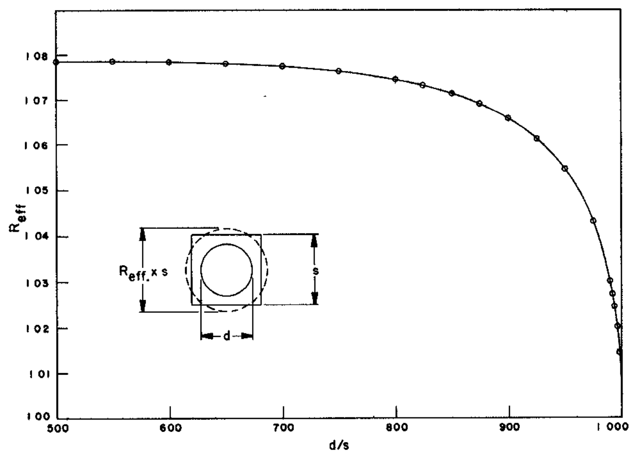

Fig. 5. Graph of R_{eff} versus d/s.

TABLE I
Values of R_{eff} For Given d/s

d/s	5	6	7	8	9	95	998
Reff	1 07861	1.07831	1 07731	1 07439	1 06568	1 05443	1.01471

and can be determined from Fig. 5 with an error of the order of 0.02 Ω for impedance values greater than 3 Ω. Where an analytic expression is desired, it should be noted that formulas given by Wheeler [16, (34), (35)] agree with the results of this paper within 0.7 percent.

The accuracy of the method is dependent on the ability of the computer to invert matrices of high order. For the $d/s = 0.998$ case, it was found that the choice of R_2 was critical for this operation. It is believed, but not known, that performing the approximation in two steps may materially simplify this inversion problem.

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Theoretical and Experimental Study of the Resonant Frequency of a Cylindrical Dielectric Resonator

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***Abstract*—A rigorous modal method is described for calculating the resonant frequencies of a circular cylindrical dielectric rod placed between two perfectly conducting plates. Comparisons of the numerical results with those obtained from another rigorous theory developed at the same time by one of the authors show an accuracy better than 10^{-4}. Comparison with experimental data shows generally a very good agreement.**

I. Introduction

During the last decade, dielectric resonators met an increasing interest, due to the development of temperature-stable materials [1], [2].

Several methods have been proposed in order to solve the problem of determining resonant frequencies. The earliest papers dealt with simple devices like a sphere, or a cylinder between metallic planes [3]–[5]. Practical devices require more intricate calculations, and solutions are often approximate [6]–[11].

The theoretical method described here can be considered as an extension of the rigorous study by Hakki and Coleman [4] in the case where the distances between the dielectric rod and the metallic plates are different from zero. Space is divided into two complementary cylindrical regions where the field is expressed in the form of two modal expansions with unknown coefficients. The matching between these two expansions leads to an infinite set of homogeneous linear equations. The resonant frequency is obtained by looking for the zero of the determinant of the truncated matrix.

A comparison is made with another rigorous theory, the differential theory, quite different in nature, and the relative discrepancy never exceeds 10^{-4} when the two methods can be implemented. Even though the same conclusion cannot be drawn for the comparison with experimental data, the agreement is very good and appears to be satisfactory, taking into account the uncertainties about the actual experimental parameters and the influence of the finite conductivity of the two plates. Thanks to the great precision of the computer code, we are able to show that the resonant frequency may be estimated very simply from an equivalence rule, provided the air gaps are small.

II. Theory

A. Basic Equations

We deal with the circular cylindrical rod represented in Fig. 1, with permeability μ_0, real relative permittivity ϵ, length h_2, and radius R. It is placed at distances h_1 and h_3 from two perfectly conducting plates parallel to the Oxy plane. We denote by $l = h_1 + h_2 + h_3$ the distance between these two plates.

The aim of this study is to compute the fundamental TE resonant frequency. The θ component $F(r, z)$ of the electric field, which is independent of θ, satisfies the following equation:

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \left(k^2(r,z) - \frac{1}{r^2}\right)F + \frac{\partial^2 F}{\partial z^2} = 0 \tag{1}$$

where $k^2(r, z)$ is the wavenumber which is equal to $k_0^2 = (2\pi/\lambda)^2$ in the air and to $k_0^2\epsilon$ in the rod.

On top of that, F must satisfy the following boundary conditions:

$$\begin{aligned} &F(r,0) = F(r,l) = 0 \\ &F \text{ and } \frac{\partial F}{\partial z} \text{ are continuous for } z = h_1 \text{ and } z = h_1 + h_2 \\ &F, \frac{\partial F}{\partial r} \text{ and consequently } \frac{\partial (rF)}{\partial r} \text{ are continuous for } r = R. \end{aligned} \tag{2}$$

Of course, $F(O, z)$ must vanish since F is a θ component, and $F(r, z)$ satisfies a radiation condition when $r \to \infty$; in other words, the field must decay exponentially outside the resonator.

B. Modal Expansions

Space is divided into two regions Ω_{ext} $(r > R)$ and Ω_{in} $(r < R)$. For both regions, we establish that the field may be expanded in series.

In Ω_{ext}, $k^2(r, z)$ remains constant and equal to k_0^2. From (1)

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